UDC 539.3

ON THE BENDING OF A PLANE INHOMOGENEOUS CURVED BEAM

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The plane problem of pure bending of a uniformly inhomogeneous curved beam bounded by two arcs of concentric circles and two radii, is considered. It is assumed that the material of the beam is isotropic, has a constant Poisson's ratio and a coordinate-dependent Young's modulus. It is shown that when the Young's modulus is defined in a specified manner, then the problem has an exact, elementary solution, and the solution is given.

1. Let us take the center of the circles $r = a$ and $r = b\,(a < b)$ as the coordinate origin, and direct the polar x-axis along the axis of symmetry of the region occupied by the beam (see Fig. 1). The angle 2α contained between two extreme radial cross sections (end faces) is assumed to be arbitrary, but smaller than 2π. The load is given in the form of bending moments $\bar{M}$ (per unit area) applied to the end faces $\theta = \pm\alpha$. The problem is studied in a linear formulation (it is assumed tha the material undergoes small deformations and obeys the generalized Hooke's Law)

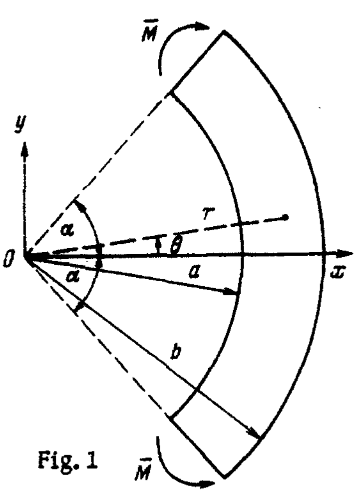

Fig. 1

$$\varepsilon_r = e(\sigma_r - \mu\sigma_\theta), \quad \varepsilon_\theta = e(\sigma_\theta - \mu\sigma_r) \qquad (1.1)$$
$$\gamma_{r\theta} = 2e(1 + \mu)\,\tau_{r\theta}$$

In the case of the gerneralized plane state of stress $e = 1/E$ and $\mu = \nu$, and in the case of a plane deformation $e = (1 - \nu^2)/E$, $\mu = \nu/(1 - \nu)$. The condition of compatibility of the deformations has the form (see [1])

$$\left(\frac{\partial^2}{\partial\theta^2} - r\frac{\partial}{\partial r}\right)\varepsilon_r + r\frac{\partial^2}{\partial r^2}(r\varepsilon_\theta) - \frac{\partial^2}{\partial r\partial\theta}(r\gamma_{r\theta}) = 0 \qquad (1.2)$$

In the homogeneous and continuously inhomogeneous beam with the elastic characteristics E (Young's modulus) or β_{ij} depending only on the distance r, the stresses are the same in all radial cross sections. The stress function depends only on r, i. e.

$$F = f(r), \quad \sigma_r = f'(r)/r, \quad \sigma_\theta = f''(r), \quad \tau_{r\theta} = 0 \qquad (1.3)$$

At the curvilinear sides we have $\sigma_r = 0$, and at the end faces σ_θ reduces to the bending moment $\bar{M}$. This yields the final three conditions (see [2], Sect. 24 and [3] ch. III)

$$f'(a) = f'(b) = 0, \quad f(b) - f(a) = \bar{M} \qquad (1.4)$$

2. Let us formulate the problem as follows: 1) to establish what form the function

$e\ (r, \theta)$ must assume for the stresses in the beam in question to have the form (1. 3), and 2) to find the corresponding function f' and the stress components. To do this, we must find the solutions of the equation (1. 2), i. e. the functions $e\ (r, \theta)$ and $f'\ (r)$ and to fulfil the conditions (1. 4).

We can rewrite the equation (1. 2) as follows:

$$\left(\frac{f'}{r} - \mu f''\right)\frac{\partial^2 e}{\partial\theta^2} - r\frac{\partial}{\partial r}\left[e\left(\frac{f'}{r} - \mu f''\right)\right] + r\frac{\partial^2}{\partial r^2}\left[e\left(rf'' - \mu f'\right)\right] = 0 \qquad (2.\ 1)$$

The function e must be positive at all points r, θ of the region occupied by the beam since $E > 0$, and we allow e to vanish $(E = \infty)$ only at the end faces $\theta = \pm\alpha$ of the beam.

We shall not analyze the problem in its entirety, restricting ourselves only to indicating the class of particular cases of inhomogeneity for which the exact solution can be found by elementary methods. We shall seek the solution of (2. 1) in the form of a product

$$e = R\ (r)\Phi\ (\theta)$$

After substitution into (2. 1), the variables can be separated and we obtain two equations

$$\Phi'' + n^2\Phi = 0 \qquad (2.\ 2)$$

$$f^{\mathrm{IV}} + 2\left(\frac{R'}{R} + \frac{1}{r}\right)f''' + \left(\frac{R''}{R} + \frac{2-\mu}{r}\frac{R'}{R} + \frac{\mu n^2 - 1}{r^2}\right)f'' - \left(\frac{\mu}{r}\frac{R''}{R} + \frac{1}{r^2}\frac{R'}{R} + \frac{n^2-1}{r^3}\right)f' = 0 \qquad (2.\ 3)$$

where n is real, pure imaginary or zero. Clearly, Φ is expressed in terms of the elementary functions and f' can be determined as a general integral of a linear, third order equation with variable coefficients. Three constants entering this integral can be found from the conditions (1. 4). In this manner we obtain an expression for the elastic characteristic e for which the stresses have the form (1. 3)

$$e = R\ (r)\ (A\cos n\theta + B\sin n\theta), \quad n \neq 0 \qquad (2.\ 4)$$

$$e = R\ (r)\ (A + B\theta), \quad n = 0 \qquad (2.\ 5)$$

The constants A, B and n and the function R cannot be completely arbitrary, since we have, in all cases, $e > 0$ (except perhaps at the points lying on the end faces $\theta = \pm\alpha$).

When R is defined in an arbitrary manner, then the search for the solution of (2. 3) encounters difficulties. If however the function R is defined in the form of a power function $R = r^m$ (m is a real number), then the equation can be solved without any difficulty. The solution depends on the roots of a third degree equation obtained from (2. 3). If none of the roots s_j are multiple, then

$$f' = C_1 r^{s_1} + C_2 r^{s_2} + C_3 r^{s_3}$$

The simplest cases arise when $m = n$, $m = -n$ and $m = 0$

1) $m = n, \quad s_1 = 1, \quad s_{2,3} = -n \pm \sqrt{1 - n(1-\mu) - \mu n^2}$

2) $m = -n, \quad s_1 = 1, \quad s_{2,3} = n \pm \sqrt{1 + n(1-\mu) - \mu n^2}$

3) $m = n = 0, \quad s_1 = s_2 = 1, \quad s_3 = -1$

In the last case (formula (2. 5) with $R = 1$) the distribution of stresses in an inhomogeneous isotropic beam will be the same as in a similar homogeneous isotropic beam.

We note that the solutions (2. 4), (2. 5) are not suitable for a hollow cylinder, the transverse section of which represents a solid, uncut ring. If we assume that the elastic characteristics of the form (2.4) and (2.5) exist, it will mean that segments will exist in the region of transverse cross section in which $E < 0$, i.e. the solutions will be physically meaningless.

REFERENCES

1. L e k h n i t s k i i, S. G., Theory of Elasticity of an Anisotropic Body. Ed. 2., Moscow, "Nauka", 1977.

2. K o l c h i n, G. B. Computation of the Elements of Constructions in Elastic Inhomogeneous Materials. Kishinev, "Karta Moldoveniaska", 1971.

3. K o l c h i n, G. B. Plane Problems of the Theory of Elasticity of Inhomogeneous Bodies. Kishinev, "Shtiintsa", 1977.

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